$\boxed{1}$

Prop (Euclid's prop): If p is prime, $p \mid ab$, then $p \mid a$ or $p \mid b$.

$p \mid p \cdot p$

Lemma (Bezout's lemma): If a, b are coprime then

$$ax + by = 1$$

has a solution w/ $x, y \in \mathbb{Z}$

e.g. $6 = p$

$6 \mid 2 \cdot 3$

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__Pf__ (Euclid's prop): If $p \mid a$, we're done.

Assume $p \nmid a$. Since p is prime this implies that a and p are coprime.

By Bezout's lemma, we can solve

$$1 = ax + py$$

$x, y \in \mathbb{Z}$. But, this means

$$b = abx + pby.$$

But $p \mid \boxed{ab}x$ and $p \mid pby$. So, $p \mid b$. $\blacksquare$

§1 Mathematical induction

<u>Thm</u> (Principle of mathematical induction): Suppose a mathematical statement $P(n)$ for every $n \in \mathbb{N}$. Then, if

- (<u>Base case</u>): $P(0)$ is true,
- (<u>induction hypothesis</u>): $P(n) \Longrightarrow P(n+1)$ is true)

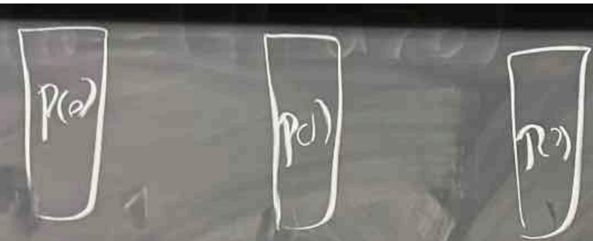
Then, $P(n)$ is true for all n.

<u>e.g.</u> $P(n)$ = "n can be written as x^2+y^2 w/ $x,y \in \mathbb{Z}$"

$P(n)$

$Q(n) = P(-n)$

4



Prop: for all $x \in [0,1]$ $\boxed{1+x \le e^x}$ $P(x)$

"Pf": Base Case: $P(0)$: $1+0 \le e^0$ $[0,1]$

Pf (of Thm): Let $S = \{ n \in \mathbb{N} : P(n) \text{ is false} \}$. Assume, $S \neq \emptyset$. By LNP, S has a minimal element $n_0 \in S$.

5.

By base case $n_0 \neq 0$. But then, $n_0 - 1 \in \mathbb{N}$. Since n_0 is minimal, $n_0 - 1 \notin S$, So $P(n_0 - 1)$ is true.

By $\boxed{IH}$, $P(n_0)$ is true. Contradiction. $\blacksquare$

Prop: For all $n \geq 1$, $\overbrace{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}^{\text{“}P(n)\text{”}}$

6.

Pf: We proceed by induction.

Base Case: The base case is $1 = \frac{1(1+1)}{2}$ which is true.

IH: Assume $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Then

$1+2+\cdots+n = \frac{n(n+1)}{2}$

$$\boxed{1+2+\cdots+n} + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

by IH. But, $\frac{n(n+1)}{2} + n+1 = \frac{(n+1)(n+2)}{2}$

$$1+2+\cdots+n+(n+1) = \frac{(n+1)(n+1+1)}{2}$$

$\blacksquare$

Pf (Gauss):

$$
\begin{array}{r}
1 + 2 + \cdots + n \\
+\; n + n-1 + \cdots + 1 \\
\hline
n\cdot(n+1)
\end{array}
$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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Prop: For all $n \geq 4$, $n! > 2^n$.

$2! = 2 \not> 2^2$

Pf: We proceed by induction.

Base Case: $4! = 24 > 16 = 2^4$ which is true.

IH: Assume $n! > 2^n$. Then

$$(n+1)! = n! \cdot (n+1) > 2^n \cdot (n+1) \geq 2^n \cdot 2 = 2^{n+1}$$

as desired. $\square$

Definition: The *Fibonacci numbers* F_n for $n \in \mathbb{N}$ are defined by the recursive formula

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2.$$

e.g. $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ ...

Prop (Cassini's identity): For all $n \geq 1$

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n$$

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Prop (Cassini's identity): For all $n \geq 1$

$$F_{n-1} \cdot F_{n+1} - F_n^2 = (-1)^n$$

$n = 2$

$F_1 \cdot F_3 - F_2^2$

$1 \cdot 2 - 1^2 = 1 = (-1)^2$

Pf: We proceed by induction

$\boxed{12}$

$$
\begin{aligned}
F_n \cdot F_{n+2} - F_{n+1}^2 &= F_n \left(F_{n+1} + F_n \right) - F_{n+1}^2 \\
&= F_n \cdot F_{n+1} + F_n^2 - F_{n+1}^2 \\
&= F_n^2 + F_{n+1} \left(F_n - F_{n+1} \right) \\
&= F_n^2 + F_{n+1} \left(F_n - (F_n + F_{n-1}) \right) \\
&= F_n^2 - F_{n+1} \cdot F_{n-1} \\
&= -\left(F_{n+1} F_{n-1} - F_n^2 \right) \\
&= -1 \cdot (-1)^n \\
&= (-1)^{n+1}
\end{aligned}
$$

$\boxed{\text{TIP}}$ **Prop:** Show that $4^n - 1$ is divisible by 3 for $n \geq 1$.

Pf: We proceed by induction.

Base Case: $4^1 - 1 = 3$ which is divisible by 3.

Inductive Step: Assume $4^n - 1$ is divisible by 3.

Then $4^{n+1} - 1 = 4 \cdot 4^n - 1 = 4 \cdot 4^n - 4 + 3 = 4(4^n - 1) + 3$

As $4^n - 1$ and 3 is divisible by 3 so is this sum $\blacksquare$

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Defn: A binary string is a sequence of 1 and 0

e.g. 01101101011

1000000110101

00010000100, 01010

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Prop: The number of binary sequences of length n w/ no consecutive 1's is F_{n+2}.

$$P(n) \Leftarrow P(n-1)$$

Pf: We proceed by strong math. induction.

Base cases: $n=1$, only 0 and 1, total # is $2 = F_3$

$n=2$, only 00, 01, 10, total # $= 3 = F_4$.

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Assume, total # of length n seq. w/ no consec. 1's is F_{n+2}. Consider a binary sequence $a_1 a_2 \cdots a_{n+1}$ w/ no consec. 1's.

Case 1: $a_1 = 0$

Then $a_2, \cdots, a_{n+1}$ can be any bin. seq. w/ no consec 1's. By SIH there are F_{n+2} of those.

Case 2: $a_1 = 1$

Then $a_2 = 0$. But, $a_3, a_4, \cdots, a_{n+1}$ can be any bin. seq. w/ no consec. 1's. By SIH the total number is $F_{(n-1)+2} = F_{n+1}$.

So, the total # across both cases is $F_{n+2} + F_{n+1} = F_{n+3} = F_{(n+1)+2}$ $\blacksquare$